Liaisons Dangereuses: Increasing Connectivity, Risk Sharing, and Systemic Risk.

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Abstract

We characterize the evolution over time of a network of credit relations among financial agents as a system of coupled stochastic processes. Each process describes the dynamics of individual financial robustness, while the coupling results from a network of liabilities among agents. The average level of risk diversification of the agents coincides with the density of links in the network. In addition to a process of diffusion of financial distress, we also consider a discrete process of default cascade, due to the re-evaluation of agents assets. In this framework we investigate the probability of individual defaults as well as the probability of systemic default as a function of the network density. While it is usually thought that diversification of risk always leads to a more stable financial system, in our model a tension emerges between individual risk and systemic risk. As the number of counterparties in the credit network increases beyond a certain value, the default probability, both individual and systemic, starts to increase. This tension originates from the fact that agents are subject to a financial accelerator mechanism. In other words, individual financial fragility feeding back on itself may amplify the effect of an initial shock and lead to a full fledged systemic crisis. The results offer a simple possible explanation for the endogenous emergence of systemic risk in a credit network.

1 Introduction

Credit contracts establish connections among commercial banks on the interbank market, among firms and banks on the market for loans, among customers and suppliers on the market for trade credit. In other words, many markets for credit can be conceived of as *credit networks* in which nodes represent agents and links represent credit relationships. This connection-enhancing property is the main reason why credit is pervasive in modern

financially sophisticated economies. Because of the risk of insolvency, the extension of credit is conditional upon the assessment of credit worthiness and the establishment of a trust/customer relationship between a borrower and a lender. This requires time and effort – there are non-negligible transaction costs in credit markets – so that an agent is willing to get into a credit contract only with few other agents: *Credit networks are therefore generally incomplete.* As a consequence in a credit network agents are generally clustered in neighborhoods so that most of the times the action (and interaction) is at the local level. Connections among neighborhoods, however, albeit sparse, make the network as a whole responsive to shocks hitting any node. The pervasiveness of credit is a mixed blessing. On the one hand establishing several credit relationships allows an agent to carry on an investment project which would not be feasible by means of internal financial resources alone and to diversify the risk of a loss (or of a credit crunch) if the agent is hit by a negative shock. In other words, connectedness allows to share the risk of an idiosyncratic shock hitting, say, agent A among her neighbors B and C. Risk sharing through the propagation of distress alleviates the burden of adjustment for agent A. On the other hand, at least in principle, distress propagation can make life more difficult for agents B and C. They may be weakened – we will be more precise in a while – by the effort to adjust to the (fraction of the) shock hitting A and transmitted through the network. Hence they may become more vulnerable to future idiosyncratic shocks. Connections among neighborhoods complete the picture, making the entire network weaker and more vulnerable. The seeds are sown for a collapse of the entire credit network in the future. In this context, in fact one cannot rule out the risk of a *systemic* crisis, i.e. not only of the diffusion but also of the *amplification* of financial distress. Each one of the interdependent agents, in fact, is susceptible to go bankrupt so that the network is potentially exposed to the risk of a systemic failure, i.e. of multiple joint bankruptcies. It is almost a commonplace, in the light of the Global Financial Crisis, that *systemic risk* has been generally neglected in the current literature. On the other hand the benefits of diversification and *risk sharing* have been almost exclusively emphasized. Both properties, however, are present and important in a credit network. In this paper we make a small step in the direction of incorporating both mechanism in a general model of a credit network. The most influential example of network analysis applied to the propagation of distress in a credit network is the seminal paper by Allen and Gale (2001) on "financial contagion". [1] The most recent strand of research has focused on three types of propagation of financial distress: (a) bank runs and financial contagion on interbank markets (Diamond and Dybvig, 1983)Allen and Gale (2001); (b) depreciation of a common asset (asset price contagion) (Kiyotaki and Moore, 1997), (Kiyotaki and Moore, 2002); (c) interlocking credit exposure (Allen and Gale, 2001), (Allen and Gale, 2005). These mechanisms represent independent but not alternative *contagion channels* which may interact during the development of a financial

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[1]See also Freixas et al. (2000) Furfine (2003); Boss et al. (2004); Elsinger et al. (2006); Iori et al. (2006) Nier et al. (2007)Boissay (2006) and Battiston et al. (2007) focus on the trade-credit relationships among downstream and upstream firms along the "supply chain".

crisis. For instance, a run on a bank is a shock which may trigger at the same time an avalanche of deposit withdrawals at other banks – i.e a propagation mechanism of type (a) – and a phenomenon of financial contagion and liquidity evaporation on the interbank market – i.e. a propagation mechanism of type (c).

Focusing on the last channel, in their pioneering contribution Allen and Gale reach the conclusion that if the credit network of the interbank market is a *credit chain* – in which each agent is linked only to one neighbor along a ring – the probability of a collapse of each and every agent (a bankruptcy avalanche involving all the agents) in case a node is hit by a shock is equal to one. As the number of partners of each agent increases, i.e. as the network evolves toward completeness, the risk of a collapse of the agent hit by the shock goes asymptotically to zero, thanks to *risk sharing*. The larger the pool of connected neighbors whom the agent can share the shock with, the smaller the risk of a collapse of the agent and therefore of the network, i.e. the higher network resilience. Systemic risk is at a minimum when the credit network is *complete*, i.e. when agents fully diversify individual risks. In other words, there is a monotonically decreasing relationship between the probability of individual failure/systemic risk and the degree of *connectivity* of the credit network.

In the present paper we adopt the same perspective of Allen and Gale, i.e. we focus on inter-linkages of credit exposures in a general framework for the analysis of credit networks. We are much less optimistic, however, on the effects of connectivity on systemic risk because, as the former increases, under specific conditions we detect the emergence of a *trade off* between decreasing individual risk – due to risk sharing – and increasing systemic risk – due to the propagation of financial distress. The larger the number of connected neighbors, the smaller the risk of an individual collapse but the higher systemic risk may be and therefore the lower network resilience. In other words, in our paper, the relationship between connectivity and systemic risk is not monotonically decreasing as in Allen and Gale, but follows a hump shaped pattern: it is decreasing if the degree of connectivity is (relatively) low and increasing when it becomes high. In particular we find that as the degree of connectivity increases above a certain threshold, crises tend to be not only more severe, but also more frequent.

This remarkable result is due to the possibly contrasting effects of two essential features of a credit network which we will label *interdependence* (of agents'financial conditions) and *financial accelerator*. *Interdependence*, which is largely predominant in the existing literature, from (Allen and Gale, 2001) to (Shin, 2008) consists in the fact that the financial condition of an agent (measured for instance by the equity ratio, i.e. the reciprocal of leverage) is affected by the financial conditions of her neighbors and thus depends on the location of the agent in the network. Interdependence is indeed a general feature of credit networks. For instance, if A lends to B, then the financial fragility of B affects the asset value of A and thus A's fragility. Interdependence can also stem from an insurance contract or a Credit Default Swap. Suppose B sells protection to A against the risk of depreciation of a security X. If the price of X actually falls and in addition B is under

distress then A's financial condition weakens because B may not be able to deliver the protection she promised.

The financial accelerator in the present setting consists essentially in the fact that a change in one agent's financial condition today may feed back on itself leading to a change in the same direction of the agent's financial condition tomorrow. This is essentially a *positive financial feedback* at the individual level over time. In other words, the financial distress of an agent today is likely to lead to additional financial distress for the same agent tomorrow. Also the financial accelerator is a pervasive feature of credit networks whose role, however, has been somehow downplayed (at least so far) in the existing network literature. The financial accelerator is activated in at least two scenarios. In the first one (see e.g. in (Morris and Shin, 2008)), an agent is hit by a shock due to a loss of market value of some securities in her portfolio. If such shock is large enough, so that some of her creditors claim their funds back, the agent is forced to liquidate at least part of her portfolio to reimburse debt. If the securities are sold below the market price, the asset side of the balance sheet decreases more than the liability side and the agent's fragility - i.e. leverage - goes up unintentionally. This situation can lead to a spiral of losses and decreasing robustness. In the wording of Adrian and Shin, leverage turns out to be pro-cyclical (Brunnermeier, 2008; Brunnermeier and Pederson, 2009). A second scenario is the one in which when the agent is hit by a shock in t, her creditors makes credit conditions harsher in t+1. Indeed it is well documented that lenders ask a higher *external finance premium* when the borrowers' financial conditions worsen (Bernanke et al., 1999). Stricter credit conditions can be considered an additional shock hitting the borrower in t+1. In a sense, both interdependence and the financial accelerator can be conceived of as *propagation mechanisms.* Interdependence allows propagation over space - i.e. from one agent to her neighbors - in the network while the financial accelerator allow propagation over time, i.e. from one agent to herself over different time periods.

Interdependence is a necessary condition for risk sharing. The higher the number of neighbors whom an agent hit by a negative shock can share the shock with, the more effective is risk sharing. On the other hand, connectedness can activate a financial accelerator. If the spreading of the "disease" is virulent enough, the risk of a systemic crisis becomes unbearably high. The interplay of interdependence and financial accelerator has not been incorporated so far in a dynamic model, although the idea is present in several previous works (Stiglitz and Greenwald, 2003, e.g. 140-141). Suppose, for instance, that agent A is hit by an adverse shock which makes her financial condition more fragile. Her lenders – say B and C – will absorb part of the shock and experience therefore a decrease of their own financial robustness (interdependence). As a consequence, they will be less willing to extend credit at the same terms as before, thus rationing the quantity of external finance made available or increasing the interest rate charged to A. In other words credit will be extended at less favorable conditions making the financial position of A even worse: Financial fragility feeds back on itself at the individual level (financial accelerator). Furthermore, A is in turn linked as a lender to other agents – say D and E.

Being in financial distress, A will restrain credit or charge a higher interest rate to her borrowers, making their financial conditions worse (interdependence again). The increase of financial fragility therefore spreads through the credit network, a phenomenon often called *distress propagation.*

An initial shock may or may not force A into bankruptcy. If the agent is financially robust, she can absorb the shock and its consequences – in particular the increase in the external finance premium – without going bankrupt. The agent, however, will be financially more fragile and more vulnerable in case a new shock occurs. If A is not "robust enough" she will go bankrupt [2] and exit, there will be a loss of "organizational capital" (in Howitt's wording) and a fall out on her partners, who will face a loss and/or incur an additional cost. The lenders (i.e. B and C) will face a loss because they will not recover the loans they extended to the bankrupt agent. The borrowers (i.e. D and E) will incur a cost because they will have to look somewhere else for credit and establish a new relation of trust with a new lender. This is an instance of transaction costs on the credit market.[3] These additional losses and costs may force some others agents into bankruptcy, and trigger a *bankruptcy cascade.*

All in all, the three mechanisms of interdependence, financial accelerator and bankruptcy cascades can be conceived of in the most general terms as externalities. In case of a negative shock to an agent, they result in additional costs to the other nodes in the neighborhood. Distress propagation is a consequence of the *interdependence* of agents in the network (it is indeed a mechanism of propagation *over space* as mentioned above). In the absence of the other two mechanisms, the probability of individual bankruptcy will tend to zero (and so will the probability of a systemic crisis) due to the propagation and therefore the sharing of distress as the connectivity and the size of the network increases. The financial accelerator changes the picture dramatically. It plays the role of a negative externality because it captures the reaction of the partners of an agent to her recent increase of fragility. Finally, bankruptcy cascades are also a negative externality because they imply that the robustness of the agent decreases when one or more of her partners goes bankrupt. Together, the three mechanisms may amplify the effect of the initial shock and lead to a full fledged *systemic* crisis if they more than offset the benefit of risk sharing. In other words a systemic crisis may originate in a single node of the network due to financial contagion and the positive feedback mechanism. It is not necessarily associated to an *aggregate* economy-wide shock.

In a broader perspective, this conceptual framework may have far reaching implications also for the assessment of the costs and benefits of globalization. Since some credit relaions involve agents located in different countries, national credit networks are connected in a world wide web of credit relaionships. The increasing inter-linkage of credit networks –

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[2]Notice that even if the shock were relatively small it could lead to bankruptcy due to the amplification mechanism discussed above.

[3]As a matter of fact, also B and C, i.e. the lenders of the bankrupt agent, have to look for some other borrowers whom to lend to. This means that they too will incur transaction costs in case of default of A.

one of the main features of globalization – allows for *international risk sharing* but it also makes room for the *propagation of financial distress* across borders. The recent, and still ongoing, financial crisis is a case in point.

International risk sharing may prevail in the early stage of globalization, i.e. when connectivity is relatively "low". An increase in connectivity at this stage therefore may be beneficial. On the other hand, if connectivity is already high, i.e. in the mature stage of globalization, an increase in connectivity may bring to the fore the internationalization of financial distress. An increase in connectivity, in other words, may increase the likelihood of financial crises worldwide.

The paper is organized as follows. In section 2 we present a simple model of the evolution of individual financial robustness. Section 3 is devoted to a definition of the issues we want to investigate, i.e. the probability of bankruptcy and systemic risk. In sections 4 we present the baseline scenario, in which only risk sharing and the distress propagation effects are considered. In section 5 we consider the additional effect of financial accelerator and in section 6 we examine the implications of the bankruptcy cascade effect. Section 7 concludes.

2 The General Set-up

In this section we derive a minimal dynamic model of the financial robustness of individual agents connected in a network of credit relationships. This formulation of the model includes the mechanisms of interdependence and financial accelerator, while the bankruptcy cascade mechanism will be introduced in Section 4. Similarly to (Hull and White, 2001), financial robustness of agent i – measured for instance by the equity ratio – plays the role here of an indicator of the agent's creditworthiness or distance to default, and denoted as $\rho_i \in [0, 1]$, with $\rho_i = 0$ indicating bankruptcy. We obtain a system of stochastic differential equations (SDE) describing the evolution in time of all the ρ_i. The interest of this approach is that it is then possible to derive some results on the expected first passage time at 0 and thus on the probability of default of an agent taking into account the dynamics of the other agents. The SDE approach allows to incorporate in a natural way the first two mechanisms mentioned in the Introduction, namely the interdependence of financial conditions and the financial accelerator and to investigate the effect of their interplay for systemic risk.

Before going into the details of the model, we describe the network and the related definitions that will be relevant in the following. We consider a set of n agents connected in a network of financial contracts. Formally, the relations in the network are described by a graph $G = (V, E)$, where V is a set of nodes representing the agents and E is a set of directed edges representing the contracts[4]. The graph is associated with an *adjacency matrix* A where $A_{ij} = 0$ if there is no edge from i to j and $A_{ij} = 1$ if there is an edge

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[4]For an introduction to networks see, e.g., (Vega-Redondo, 2007)

from i to j – meaning that agent i owns liabilities of j.[5] The graph is also associated with a *weight* matrix W with $W_{ij} \in [0,1]$ and $W_{ij} > 0$ iff $A_{ij} = 1$. For our purposes W is a row-stochastic matrix ($\forall i \sum_j W_{ij} = 1$) where W_{ij} represents the exposure of i to j relative to i's portfolio of exposures. The *out-degree* of node i is the number k_i of out-going edges of node i and represents the number of counterparties or *neighbors* to which agent i is exposed.

In the following we will characterize the systemic risk of the credit network for varying levels of risk diversification. We are interested in the situation in which the graph is connected (i.e. there is a path from any agent to any other agent) and the exposures in the portfolio of each agent are approximately balanced, i.e. $W_{ij} \cong 1/k_i$ so that the degree of the node i is a rough measure of the risk diversification of the agent (we are not considering the optimization of the portfolio over time, therefore this measure will suffice for the results we aim to show here). The ratio of the number ℓ of existing edges among nodes over the number of all possible edges $n(n-1)$ is the *density* d of the network, $d = \ell/(n(n-1))$. It represents the fraction of possible pairs of agents that are involved in a financial contract. A simple relation of proportionality holds between the the average degree k in the network and the density $d = k/(n-1) \cong k/n$ (for n in the range we are interested in). Thus for a fixed number n of agents, the density of the network is also a measure of the average risk diversification across the agents.

The first property of our model we want to focus on concerns the fact that the current trend in the financial robustness of one agent depends procyclically on the past trend. Such mechanism, which we call *financial accelerator*, can be justified, for instance, in terms of business partners applying tighter credit conditions. This assumption is in line with the vast literature on the financial accelerator (Bernanke et al., 1999) in which the external finance premium increases when the borrower's net worth goes down, thus increasing the chances of a further decrease of net worth in the next period. In the context of a SDE describing the evolution of the robustness of the firm, the financial accelerator can be modeled as a drift term that depends on the past realizations of the robustness[6]. For the sake of simplicity, we restrict the dependence at time t on the past only to time t' (with $t' < t$). To remain general, we consider the function $h(\rho(t), \rho(t')) \leq 0$, so that the law of motion of robustness is described by the following *time-delayed* stochastic differential equation,

$$d\rho_i = h(\rho_i(t), \rho_i(t'))dt + \sigma d\xi_i, \qquad (1)$$

where σ measures the variance of idiosyncratic shocks hitting firm i and $d\xi_i$ denotes the Wiener process. As we will show in Section 3, some functional forms of h allow to remove the time delay via an analytical approximation. For the moment, we just recall a known property of the process of Equation (1). In absence of financial accelerator

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[5]The correspondence of the direction of the edge to the direction of the financial tie is only a matter of convention and does not affect the results. The convention chosen here is more natural for matrix operations.

[6]For an introduction to SDE see, e.g., Gardiner (2004); Oksendal and Karsten (1998)

$(h = 0)$, the expected first passage time T_f at 0 is $T_f \sim 1/\sigma^2$. In presence of financial accelerator ($h < 0, \exists t$), the negative drift makes the expected first passage time shorter: $T_f(h < 0) < T_f(h = 0)$. Quite expectedly, the financial accelerator, modeled as in Equation (1), tends to increase the probability of default of the individual firm, with respect to the case without accelerator. However, it is not clear at all a priori how this probability will be affected by a higher level of risk diversification.

The second property concerns the dependence of financial robustness of an agent on the robustness of the agents connected with her by some contract. As in the models of Eisenberg and Noe (2001) and Shin (2008), we consider a set of agents connected in a network of obligations. We denote as K_{ij} the asset held by i and related to j. To be concrete, let us think of this as a liability of agent j to i (other relevant situations were mentioned in the Introduction). Then, the value of K_{ij} depends on the ability of j to meet the obligation and thus on her financial robustness. We assume that the value of the asset is proportional to the robustness of the debtor, $K_{ij} = K_{ij}^0 \rho_j$ where K_{ij}^0 is the nominal value of the contract. Let us consider a discrete time, first. Then, the value of total assets of i at a given time must be determined based on the estimate of robustness already available to the agents,

$$K_i(t+1) = \sum_j K_{ij}(t+1) = K_i^0 \sum_j W_{ij} \rho_j(t), \tag{2}$$

where K_i^0 is the capital invested in the assets and W_{ij} are the fractions of capital i has invested in each contract. Thus, $\sum_j W_{ij} = 1$, where W_{ij} measures the relative initial value of the debt of j for i. In case j is not able to meet her obligation, W_{ji} is the relative loss of i with respect to her initial investment. Notice that this may differ from the loss with respect to the current value of the assets. If the returns on the various assets do not differ too much, or one considers periods of time that are not too long, W_{ij} is a reasonable proxy of the relative impact on i's asset due to a change in robustness of j.

Since i's capital depends on the robustness of the neighbors, so does equity $E_i = K_i - L_i$, where L_i is total liability of i. In particular, at constant L_i, equity increases with increasing partners' robustness and viceversa. Now, deriving a fully fledged law of motion of interdependent values of equity or any other robustness indicator would require to model also the decisions agents make regarding their capital structure and in particular whether and how to reduce their liability. Our aim here is more modest. We make the assumption that, in absence of bankruptcies, robustness values are simply linearly dependent and we explore the consequences of such an assumption on the systemic risk in a dynamic setting. We thus write

$$\rho_i(t+1) = \sum_j W_{ij} \left(\rho_j(t) + \sigma \xi_j(t) \right), \tag{3}$$

where we assume that robustness is subject to idiosyncratic identically distributed and

normal shocks[7]. Since the liabilities of agent j are held by other agents, the idiosyncratic shock hitting j is shared with the connected agents proportionally to the relative weight of their financial exposure to j. Moreover, the shock hitting agent i affects i herself only proportionally to the term W_{ii}, while she is affected by the shocks hitting the neighbors. The change in time of robustness is then

$$\rho_i(t+1) - \rho_i(t) = \sum_j W_{ij}\rho_j(t) - \rho_i(t) + \sigma \sum_j W_{ij}\xi_j(t) \tag{4}$$

Passing now to the limit of continuous time and including the term h introduced in Equation (1) to capture the financial accelerator, we obtain the linear SDE system

$$d\rho_i = \left[\sum_j W_{ij}\rho_j(t) - \rho_i(t) + h(\rho(t), \rho(t'))\right] dt + \sigma \sum_j W_{ij}d\xi_j, \tag{5}$$

where each $d\xi_j(t)$ denotes an independent Wiener process. Notice that Eq. (3) implies that the robustness of an agent is not the result of a simultaneous computation of robustness of all agents. Instead, it is affected by the change in robustness of the neighbors at the previous period. This is similar to what happens in the professional practice of financial rating. The rating of an institution is revised *after* the occurrence of events regarding the capital structure of the agent herself or the agents to which she is exposed. Since the linear combination of uncorrelated Wiener processes is still a Wiener process, denoted as $dz_i = \sum_j W_{ij}d\xi_j$, we can write

$$d\rho_i = \left[\sum_j W_{ij}\rho_j(t) - \rho_i(t) + h(\rho(t), \rho(t'))\right] dt + \sigma dz_i. \tag{6}$$

Bankruptcy occurs when robustness falls below a given threshold, which we model, as usual, as a lower absorbing barrier. Therefore, the agent goes bankrupt when robustness hits the lower barrier and is replaced by a new agent, with a new initial value of robustness. We assume that increasing financial robustness has an opportunity cost for the agent, so that she has no incentive to increase robustness indefinitely. For the sake of simplicity, we model this by assuming that robustness cannot exceed an upper barrier at $\rho_i = 1$.

According to the standard approach in finance the valuation of corporate liabilities of a firm is related to a first passage problem in a stochastic diffusion process (Black and Cox, 1976; Merton, 1974). The asset value is assumed to evolve over time as an SDE with a lower absorbing barrier, so that the firm defaults when asset value becomes zero or hits from above a positive threshold (Longstaff and Schwartz, 1995). When several firms are considered, the structural models (Hull and White, 2001; Zhou, 1997) assume that the

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[7]Within the same framework one could also account for the case of correlated shocks. For our purposes this case will suffice here.

diffusive terms of the different firms bear some mutual correlation. In particular, Hull and White (2001) have introduced a *credit index* describing the creditworthiness of the firm. The credit indices of different firms evolve as correlated stochastic processes with zero drift, but with an absorbing barrier that is time dependent and specific to each firm. In this way the authors are able to compute numerically the probability of default of large numbers of firms. Differently from those works in finance, our aim here is not to value corporate liabilities. Instead, we aim at understanding the interplay of interdependence and financial accelerator in situations of systemic risk. In this paper, we will restrict our focus on the case of drift and diffusion which are independent of the state variable ρ. However, it is possible to try and extend the results along the lines of the stochastic capital theory introduced in (Brock et al., 1989).

3   A specific model

In order to analyze the properties of this general framework, we need to specify some of its components. First of all, we need to be specific about the structure of the credit network. We assume that this is a regular graph, i.e. each agent has the same number k of partners. This is of course a strong assumption. However, our focus in this paper is on the role of the network density. Moreover, our results can be extended to random graphs in good approximation. The role of degree heterogeneity will be investigated in future work.

We need also to be specific about the function h in Equation (6). We assume that the time delay is small, $t' = t - dt$ and we adopt the following definition.

$$h(\rho(t), \rho(t - dt)) = \begin{cases} \alpha \text{sign}(\rho_i(t) - \rho_i(t - dt)) & \text{if } \rho_i(t) - \rho_i(t - dt) < -\epsilon \frac{\sigma}{\sqrt{k}} dt \\ 0 & \text{otherwise} \end{cases} \qquad (7)$$

The parameters ϵ and α represent, respectively, the *sensitivity* and the *amplitude of the reaction* of the neighbors of i to a decrease in her robustness. The definition above implies that the neighbors of i react only when an adverse shock to i is large enough to cause a negative variation of robustness that exceed, in absolute value, ϵ times the standard deviation of the shocks (which is $\sigma/\sqrt{k}$). When the neighbors do react, the reaction causes a decrease of magnitude α in the i-th robustness (because of the additional costs imposed to i by the neighbors, as explained in the Introduction). The most important aspect of this formulation is that the sensitivity of agents' response depends on the density of the network, k. More dense networks lead to more sensitive responses (in the sense that agents respond at a smaller threshold on the signal). The basic rationale for this is that for a fully diversified firm, "poor" performance means a relatively less severe loss than "poor" performance for a firm with more concentrated portfolio. Indeed in presence of greater concentration, few bad investments can lead to financial distress, since they each make up a larger percentage of the firm's overall portfolio. Neighboring firms are

assumed to recognize this and adjust their distress response levels accordingly. In a full general equilibrium context this will arise because more diversified firms will take on higher levels of debt financing, which will offset the benefits of greater diversification. It also occurs because (i) more dense networks are inherently less transparent at the level of the individual firm (more investments are more difficult to monitor) and (ii) the signal implied by poor results at one firm has stronger impact for the likelihood that others are in trouble if the degree of interconnectedness is greater. Both factors will lead neighboring firms to respond more sensitively to "poor" reported results by any single firm with whom they do business.

Notice that, defined as in Equation (7), the value of h at time t is independent on the realization, at the same time t, of the Wiener process dz_i in Equation (6). In fact, $\rho_i(t) - \rho_i(t - dt) \sim d\rho(t - dt)$, in the definition of h, depends on the realization of dz at times $t - dt$ and $t - 2dt$. Whether h equals $-\alpha$ or not, obviously depends on the amplitude, $\sigma/\sqrt{k}$, of the noise and on the parameters ϵ and α. However, $h(t)$ is a stochastic variable, independent of dz at time t. It is $h(t) = -\alpha$ with a certain probability, say q, and the average value over time of the variable h is then $-\alpha q$, which corresponds to a constant negative drift. Since we are interested in the average first passage time of our system, from now on we model the financial accelerator simply as a constant drift term and rewrite Equation (6) as follows

$$d\rho_i(t) = \left(\sum_{j=1}^{k_i} W_{ij}\rho_j(t) - \rho_i(t) - \alpha q(k, \sigma, \alpha, \epsilon)\right) dt + \frac{\sigma}{\sqrt{k}}dz_i. \tag{8}$$

Notice that q depends on the degree k of diversification. The expression of the probability q will be given in Section 3.1.

Since the evolution of robustness in Eq. (8) is described by a linear system, the distribution $p(\rho_i(t))$ of the values of robustness of a given node i across different realizations of the process is Gaussian (Gardiner, 2004). We cannot say the same for the probability distribution $p(\rho, t)$ of the values of robustness across nodes in a given realization at a given time. The two distribution are not the same a priori. However, via the Ito lemma one can derive a scalar SDE for the variance $v(t) = \sum_j (\rho_j(t) - \bar{\rho}(t))^2/N$ of the robustness across the nodes at any time[8]. Under some mild condition [9] the expected value of the variance tends exponentially fast over time to the value $\frac{\sigma^2}{k}(1 - \frac{k}{N})$. Therefore, apart from the correction term $-\frac{k}{N}$ due to the combination of shocks, as soon as there are enough links in the network, the variance of the robustness across agents coincides with the variance of the shocks at the level of individual agents. Over time the trajectories of $\rho_i(t)$ at the different nodes evolve more and more closely as k increases. Therefore, it is reasonable to

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[8]This is a standard application of the Ito Lemma (Oksendal and Karsten, 1998). A proof is anyway available from the authors upon request.

[9]The graph has to be primitive (Seneta, 2006).

assume in the following that, over time, the distribution of robustness across agents can be approximated by a Gaussian with variance $\sigma/\sqrt{k}$.

3.1 The Probability of Failure

In this section we analyze the impact of interdependence and the financial accelerator on the probability of bankruptcy. In a nutshell, we find (not surprisingly) that, in the absence of financial accelerator, the diversification of individual risk always yields a reduction of systemic risk. In contrast, as soon as we include the financial accelerator, this is not necessarily true. The financial accelerator generates a negative drift in the stochastic trajectory of the robustness that more than offsets the positive effect of diversification. The probability of individual bankruptcy can be investigated in terms of the first passage time of the SDE and it is possible to compute analytically the mean passage time at 0 and hence the probability of bankruptcy.

The probability P_f that at any given time t an agent goes bankrupt is the expected frequency with which, over time, the robustness of the agent hits the bankruptcy threshold. Since frequency is the inverse of the time between successive events, the probability of bankruptcy can be measured as the inverse of the *mean first passage time*, T_f, of the stochastic process describing the evolution of the robustness, $P_f = \dfrac{1}{T_f}$.

Proposition 1. *Consider a regular network of financial agents whose robustness follow Equations (7)(8). Denote the average probability of bankruptcy of an agent as P_f. Then:*

1. *In absence of financial accelerator ($h = 0, \forall t$), the average probability of bankruptcy of the individual agent decreases with the degree k of risk diversification as $P_f(k) = \sigma^2/k$.*

2. *In presence of financial accelerator ($h < 0 \, \exists t$) it holds:*

   (a) *depends on k as follows*

   $$P_f = \frac{\alpha q(k)}{1 + \frac{\sigma^2}{2k\alpha q(k)}(exp(-2k_i\alpha q(k)/\sigma^2) - 1)} \tag{9}$$

   with

   $$q = \frac{\Phi(-\epsilon)}{1 - \Phi(\frac{\alpha\sqrt{k}}{\sigma} - \epsilon) + \Phi(-\epsilon)} \tag{10}$$

   where Φ is the Gaussian cumulative distribution.

   (b) *$P_f(k)$ as a function of k has a unique minimum if $\epsilon = 1$, $(\alpha, \sigma) \in [0, 1] \otimes [0, 1]$ and $\alpha \leq \sigma$.*

The first result stated in the proposition is due to the fact that in absence of financial accelerator the evolution of the robustness can be approximated with a Brownian motion. In this case, the mean first passage time through an absorptive barrier is inversely proportional to the variance of the random steps. This implies the intuitive result that the probability of bankruptcy decreases with the average number of counterparties. In other words in absence of financial accelerator, the credit network is more stable the larger is the number of contracts among agents. This results is in line with the classical result of (Allen and Gale, 2001).
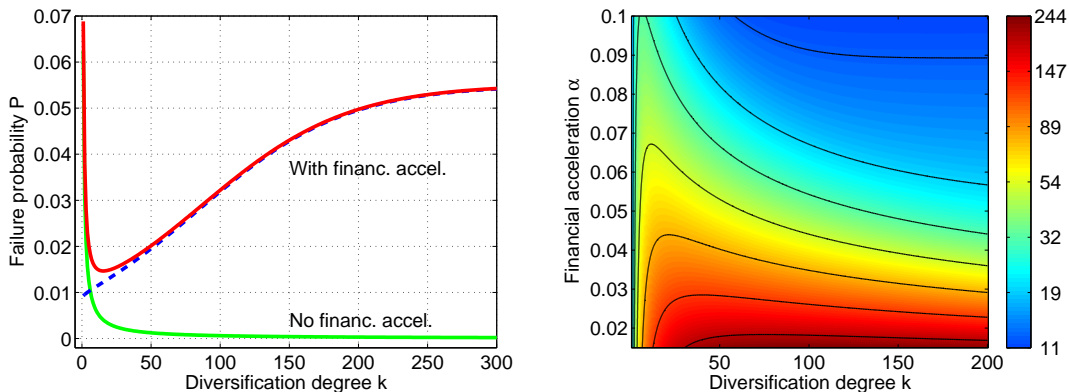

Figure 1: (Left) Failure probability P_f as a function of the diversification degree k, for a particular choice of the parameters: $\epsilon = 1$, $\sigma = 0.25$, $\alpha = 0.055$. Red and green lines refer to the process in presence and absence of the financial accelerator, respectively. The blu dashed line indicates the values of the drift $\alpha q(k)$. (Right) Expected first passage time at 0, T_f, plotted in color code as function of diversification degree k and intensity of the financial acceleration α. For readability only a zoom is shown. Lines in black represent isoclines.

For the second and third result, the dependence of q from k is shown in Figure 1. The dashed curve corresponds to the drift term $\alpha q(k)$. This means that in the stochastic process of Equation (8) the drift is negligible for small k, becomes important for large k and tends to a finite asymptotic value for $k \to \infty$. The mean first passage time of a stochastic process in presence of both a *reflective* and an *absorbing* barrier can be computed with standard approaches (Gardiner, 2004) The result is that we can write a general expression for the probability of failure $P_f(k)$ and that this function turns out to have a minimum as a function of k, in a whole range of values of α and σ. This implies that risk diversification becomes at some point counterproductive and increases the probability of failure, something which is now at odd with the result of (Allen and Gale, 2001).

This counter-intuitive result depends obviously on the fact that the financial accelerator mechanism generates a drift. How general is this result? Let us emphasize again that

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we did not assume how the drift varies with k, but that we obtained as a result of the interaction of the agents in a network. It is an emerging property.

The crucial assumption for that to be true is that the reaction of the counterparties of an agent has constant size α while the amplitude of the observed signal (the fluctuations of the shared shocks on robustness) is $\sigma/\sqrt{k}$ which decreases with k. The generality of the result depends on the generality of this assumption. Now, there is not much to question about risk diversification decreasing the fluctuation amplitude. About the amplitude of the reaction there is instead more room for objections. One could model the amplitude of the reaction to a large decrease in robustness of i as a function of ρ_i. For instance, the reaction $\alpha(\rho_i)$ could be small if i is still financially robust and become larger when her situation deteriorates. This would tend to delay the onset of the financial accelerator and hence to shift the minimum of the curve of $P_f(k)$ to the right.

While it is possible to argue that the reaction of counterparties involves actions of discrete nature which cannot be tuned to the size of the signal, a final word on this issue requires an empirical validation.

4 Bankruptcy Cascades

In the previous section we have analyzed the case in which the robustness of a firm is affected over time by the change in robustness of the connected firms. In this section we analyze the case in which, in addition, at the moment of the bankruptcy of a neighbor, the firm is further affected negatively, due to organizational loss or other effects, as discussed in the Introduction. If the firm is in turn pushed into bankruptcy, this can trigger a cascade process in which bankruptcies generates more bankruptcies. This is commonly referred to as domino effect. Similar perspective has been investigated by (Eisenberg and Noe, 2001) in interbank credit networks and by (Battiston et al., 2007) in trade credit networks. More recently, after the turmoil of the ongoing financial crisis it has been investigated by several other authors including, notably, the model of (Shin, 2008) which builds on the one of (Eisenberg and Noe, 2001). However, differently from (Shin, 2008), here we study bankrutpcy cascades in a dynamic setting, that is in combination with the time evolution of robustness and the diffusion of financial distress. The investigation of the determinants of systemic risk in such a dynamic setting has not been addressed so far. It is of course very interesting but also somewhat cumbersome to treat analytically. Here we propose a method to estimate the *probability of large cascades* P^c, that is the probability of occurrence of cascades involving a fraction of the system larger than $c \in [0, 1]$.

In our model the *bankruptcy cascade* mechanism is activated only in case of bankruptcy of one or more agents in the system, as opposed to the distress propagation effect which is at work at every time. In order to capture this effect and investigate its implications, we need to introduce a discrete cascading process. For the sake of simplicity we assume that this process is much faster than the evolution of robustness described by Equation

(6), so that we can decouple the two processes. The rationale is that distress propagation take place on the scale of the quarters (when firms issue financial statements) while a cascade can occur in a few days. For instance, the default of an important institution can determine a fast process of writing downs and reevaluation of assets among the counterparties with possible further defaults. We capture this idea by modeling, in case one or more bankruptcies occur at time t, the following recursive process taking place within the interval $[t, t + dt]$, for all i. We denote the time scale of the cascade process with the discrete variable $\tau = 1, ..., n_\tau$ At each time step τ, the variables of all agents are updated in parallel according to

$$\rho_i(\tau) = \rho_i(t) - \frac{a}{k} \sum_j W_{ij} \chi_j(\tau) \tag{11}$$

where the function $\chi_j(\tau)$ indicates if the agent j has gone bankrupt at any of the steps $1, ..., \tau$:

$$\chi_j(\tau) = \begin{cases} 1 & \text{if } \rho_j(\tau') < 0, \ \exists \tau' \leq \tau \\ 0 & \text{else} \end{cases}$$

The initial value of the process is set as $\rho_i(\tau = 1) = \rho_i(t)$. At the end of the cascade process we update the robustness as $\rho_i(t) = \rho_i(\tau = n_\tau)$.

In other words, Equation (11) simply states that the bankruptcy of one or more borrowers of i leads to a reduction of agent i's robustness. The extent of such reduction could be modeled in several ways (see e.g. Eisenberg and Noe (2001)). Here we have chosen a very simple one. The parameter a determines the extent of the damage caused by the bankruptcy of j to the neighbors as a whole. Since agent j is one out of k neighbors and the financial exposure is assumed to be evenly allocated, j represent a fraction $1/k$ of i's asset value. The damage transferred to agent i depends on the ratio a/k. Thus the value $a = 1$ corresponds to the extreme case in which all assets of the failing firm are lost.

The cascading process ends after a finite number of steps smaller than the number of agents. Notice that if the network has *small world* properties (Vega-Redondo, 2007, page 54), the number of steps is of order of $\log(N)$ and therefore very small compared to N. In the final state, a certain fraction s of the agents are bankrupt. Notice also that while the financial accelerator effect on firm i depends on i itself, here the externality on i resulting from the bankruptcy of j depends only on the weight that j represented for i's business and on the capability of agent i to recover part of the debt of j.

In order to evaluate the relationship between systemic risk and the density of the network, we first analyze the process of cascades (11) alone, separated from the diffusion of financial distress (8).

4.1 The size of a cascade

The number of failures at the end of the cascade process can be computed easily under some simplifying hypotheses. The technique consists in deriving a fix point equation for

the cumulative fraction of failures in the system as a function of the cumulative fraction of failures at the previous step. Solving for the stable fixed point of such equation, yields
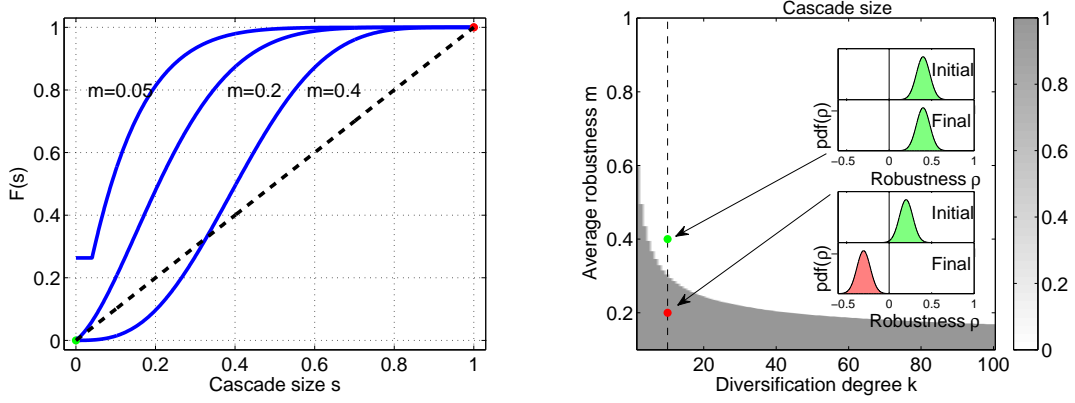

Figure 2: (Left) Plot of R.H.S. of the fixed point Equation (12) for the cascade size. Examples for $\sigma = 0.25$ and $k = 10$. For $m = 0.05$ and $m = 0.2$ the only stable fixed point is $s = 1$ (complete collapse). For $m = 0.4$ there are two stable fixed points, but the initial fraction of failures is almost zero, so the dynamics can only reach the left stable fixed point. (Right) The fraction s of failures is plotted in gray scale as a function of diversification degree k and average robustness m.

the size of the cascade. The result is stated in the following proposition. Notice that the fraction s includes the failures at the beginning of the period t, plus those occurring during the period t.

Proposition 2. *Consider the process of Eq. 11 with time variable τ. Assume the network of firms is a regular graph with degree k. Assume also the initial probability distribution of robustness is Gaussian with mean m and variance σ_ρ^2, $p(\rho, \tau = 0) \sim Gauss(m, \sigma_\rho)$. Let Φ_{m,σ_ρ} denote the cumulative probability distribution and s_0 denote the fraction of firms whose robustness is below zero at the beginning of the process. Then:*

1. *The fraction s of failures generated in the cascade process is the solution of the equation*

$$s = \max\{s_0, F(s, m, \sigma_\rho)\}, \quad where \tag{12}$$

$$F(s, m, \sigma_\rho) \sum_{j=1}^{k} \binom{k}{j} s^j (1-s)^{k-j} \Phi_{m,\sigma_\rho}(j\frac{a}{k}) \tag{13}$$

2. *A stable fixed point always exists.*

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A similar computation can be carried out for any other probability distribution of robustness $p(\rho)$. The hypothesis of a Gaussian distribution is of course a simple one, but it is also specifically relevant to our context for the reasons discussed in Section 3.

As an example, Figure 2 (Left) shows a plot of the R.H.S. of Equation (12), for some specific parameters. The proof rests on the assumption that the failures of the neighbors of a given node are statistically independent, so that we can describe the probability of multiple failures among the neighbors of a node with a binomial distribution. This assumption is strictly valid only when there are few failures in the system. When there are instead many failures in the system, the cascade process will very likely involve the whole network, independently of the correlation between neighbors' failures (as confirmed below). Therefore the computation is reasonably accurate in both cases.

For our purposes, it is convenient from now on to consider the variance of individual shocks σ as constant and the diversification k as a varying parameter, with a resulting variance $\sigma/\sqrt{k}$ for the robustness. The dependency of s on the parameters k and m, for $a = 1$, is illustrated in Figure 2 (Right). Indeed, Proposition 2 implies that there are two regimes, one with small cascades and one with large cascades. If the initial distribution of robustness has a mean m close to 0 and a large variance (small k), then obviously many firms are bankrupt already at the beginning, when the cascade process starts (see lower inset of the Figure). They affect other firms, many of which have also small robustness since the mean of the distribution is low, thus the resulting cascade is large. On the contrary, if the distribution has very small variance (large k), or large enough m (such that very few firms are bankrupt at the beginning and only few others are weak enough to fail through the cascade effect), then the cascade stops very soon (see upper inset in the Figure).

4.2 Interplay of Robustness Evolution and Cascades. Implications on Systemic Risk

Since systemic risk is the probability that a large fraction of firms in the system fail at the same time, it is now crucial to understand how frequently the system is found in the conditions that lead to a large cascade. However, when we combine the cascading process (11) with the diffusion of financial distress (8), the dynamics resulting from the interplay of the to processes is not trivial to analyze. The diffusion process shapes the distribution of robustness which determine the size of the cascades. These in turn reshape the distribution of robustness and the evolution of robustness over time. Since the cascade is, by nature, a discrete process involving recursively the neighbors of higher order of the nodes that initially fail, it is not possible any longer to describe the temporal evolution of the robustness in terms of a system of continuous stochastic differential equations. Systemic risk can be thus seen as the result of two factors: the size of the different possible cascades, and the frequency by which the system visits the distribution of robustness that lead to such cascades.

In particular a fraction $s \cong 1$ of simultaneous failures occurs whenever the average robustness m crosses the boundary of the region of large cascades (in dark) in the phase diagram of Figure 2 (Right). Recall that the values of σ and k are constant during the evolution of $\rho_i(t)$. Thus, the expected time it takes for the average robustness to reach the value m during the evolution can be approximated by the first passage time at $\rho = m$ of a representative trajectory of robustness starting from $\rho = 1$. This approximation makes sense because of the mean reversion which keeps trajectories clustered around the average, as discussed earlier. Therefore, in this approach the probability $P_f^{(c)}$ of cascades larger than c is the inverse of the first passage time at a value m such that a cascade larger than c is triggered. The probability of large cascades estimated in this way turns out to be a non monotonic function of the risk diversification. In other words, similar to the individual failure probability even the systemic failure probability display an optimal degree of diversification beyond which a further diversification becomes detrimental. This is stated in the following proposition.
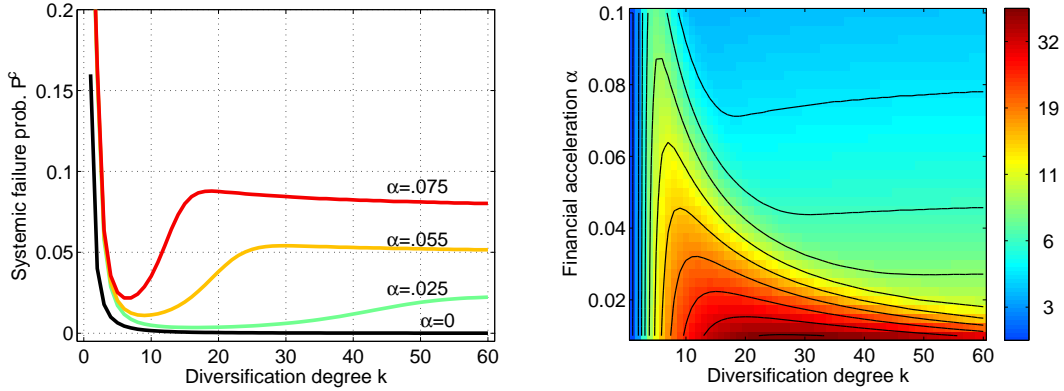

Figure 3: (Left) Probability P^c of occurrence of large cascades as a function of risk diversification k for three different values of the intensity α of financial acceleration. (Right) Expected time to large cascades T^c plotted in color code as a function of k and α. Values of the parameters: $\epsilon = 1$ and $\sigma = 0.4$ and $c = 0.25$. We vary $k = 2, ..., 100$ and α in steps of 0.001.

Proposition 3. *Consider the process of Eq. 11 with time variable τ. Assume the initial distribution of robustness is a Gaussian $p(\rho, \tau = 0) = gauss(m, \sigma)$. Assume the network of firms is a regular graph with degree k. In mean-field approximation, the probability P^c of occurrence of a systemic failure involving a fraction of the system larger than $c \in [0, 1]$, is*

$$P^c(k) = \frac{1}{T^c(k)} \tag{14}$$

where

$$T^c(k) = \min_m \{T(k,m)|S(k,m) \geq c\} \tag{15}$$

$$T(k,m) = \frac{1-m}{\alpha q} + \frac{\sigma^2}{2\alpha^2}\frac{(exp(-2\alpha kq(1-m)/\sigma^2)-1)}{kq^2} \tag{16}$$

$$S(k,m) \quad solution \ of \ Eq. \ (40) \tag{17}$$

P^c as a function of k is a non-monotonic function of k for $\alpha \in]0,0.1]$ and for at least some intervals of values of the other parameters ϵ and σ, c.

As shown by the plot in Figure 3, the probability of large cascades is not decreasing monotonically with k. On the contrary it exhibits a marked minimum for intermediate values of k. Notice that the cascade size s in Figure 2 has essentially two ranges of values: $s < 0.05$ in the regime of small cascades and $s > 0.9$ in the regime of large cascades. Therefore, the second point of the above Proposition is independent of the value c that we take as definition of "large cascade" as long as $c \in [.05, .9]$.

Finally, the results of the paper can be summarized in the following conclusive corollary.

Corollary 1 (Non-monotonicity of systemic risk). *In presence of financial accelerator, as risk diversification k in the credit network increases, systemic risk is in general not monotonically decreasing, with or without cascade effects. In particular, at least in some interval of the range of the parameters involved, there is an optimal level of risk diversification.*

5   Conclusions

In this paper, we have characterized the evolution over time of a credit network in the most general terms as a system of coupled stochastic processes, each one of which describes the dynamics of individual financial robustness. The coupling comes from the fact that agents assets are other agents' liabilities. Thus each agent's financial robustness is interdependent with the financial robustness of the counterparties. through risk sharing but also through the diffusion of financial distress as well as bankruptcy cascades.

We find that in the presence of financial accelerator, the *positive feedback* of financial robustness on itself, together with the other externalities represented by distress propagation and bankruptcy cascades may more than offset the stabilizing role of risk sharing and amplify the effects of a shock to a single node of the network, leading to a full fledged systemic crisis.

The relationship between the probability of failure both individual as well as sytemic, and connectivity is U-shaped. The stabilizing role of risk diversification prevails only when connectivity is low. If connectivity is already high, a further increase may have the

perverse effect of amplifying the shock due to distress propagation and financial accelerator.

The present work is intended to provide a basic framework and can be extended in several directions. For instance, one can study the effect of different static topological structures in the credit network, such as skewed degree distributions or degree-degree correlation (assortativity). These have been found to have prominent effects in spreading phenomena akin to technological diffusion on social networks (Jackson and Rogers, 2007). A specific and major question in our context concerns the systemic risk of a network organized in clusters which are densely connected inside but loosely connected among each other. Whether there exists an optimal level of clusterization would have important implications on the debate about the role of globalization. In addition, one can model the set of contracts among agents as an endogenously evolving network in which each agent may rewire some of her ties to the other agents if she expects to derive a higher utility or a smaller risk in doing so.

5.1 Acknowledgments

A   Appendix

Proof. Proof of Proposition 1   (1) The mean first passage time of a random walk with variance σ_r^2 is known to be $T_f = 1/\sigma_r^2$. We apply this result to our case with $\sigma_r = \sigma/\sqrt{k}$ and we obtain $P_f(k) = 1/T(k) = \sigma^2/k$.

(2) First we need to derive the expression of the probability q that the financial acceleration takes place in any given time. In the limit of small time delay we can approximate the differential, $\rho_i(t) - \rho_i(t - dt) \sim d\rho_i(t)$. Denote $q_t = P\{h_i(t, dt)\}$. The probability that $h(t, dt) = -\alpha$ can be expressed as follows.

$$q_t = P\left\{h_i(t, dt) = -\alpha\right\} = P\left\{\rho_i(t) - \rho_i(t - dt) < -\epsilon\sigma/\sqrt{k}dt\right\} \tag{18}$$

$$\cong P\left\{d\rho_i(t - dt) < -\epsilon\sigma/\sqrt{k}\right\} = P\left\{h_i(t - dt, t - 2dt) + \zeta\sigma/\sqrt{k} < -\epsilon\sigma/\sqrt{k}\right\} \tag{19}$$

where in the last step we have neglected the mean reversion term in the expression of $d\rho(t-dt)$. There are two possible cases in which the condition $h_i(t-dt, t-2dt)+\zeta\sigma/\sqrt{k} < -\epsilon\sigma/\sqrt{k}$ is fulfilled. In the first case, it is $h(t - dt, t - 2dt) = -\alpha$ (which happens with probability q_{t-dt}) and then it must be $\zeta < \alpha\sqrt{k}/\sigma - \epsilon$ which occurs according to the cdf of the Gaussian distribution, since ζdt is a Wiener process. In the second case, it

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is $h(t - dt, t - 2dt) = 0$ (which happens with probability $1 - q_{t-dt}$) and then it must be $\zeta < -\epsilon$. In formulas this reads:

$$q_t = q_{t-dt}\Phi(\frac{\alpha\sqrt{k}}{\sigma} - \epsilon) + (1 - q_{t-dt})\Phi(-\epsilon) \tag{20}$$

Solving for the fixed point in q one finds

$$q = \frac{\Phi(-\epsilon)}{1 - \Phi(\frac{\alpha\sqrt{k}}{\sigma} - \epsilon) + \Phi(-\epsilon)} \tag{21}$$

where Φ is the cumulative distribution function of the Gaussian. To show that the fixed point is stable we proceed as follows. Denote the R.H.S. of Equation (20) as $f(q) = q_{t-dt}\Phi_2 + (1 - q_{t-dt})\Phi_1$. Notice that both cumulative distributions Φ_1 and Φ_2 lay in $]0, 1[$ for finite values of the parameters. Since f is linear in q and $f(0) > 0$, a unique stable fixed point exists iff $df/dq < 1$. This is always verified because $df/dq = \Phi_2 - \Phi_1 < 1$.

Now, we use the following result (see for instance, (Gardiner, 2004), page 139). The mean first passage time of an one-dimensional homogeneous SDE

$$dx = A(x)dt + \sqrt{B(x)}d\xi \tag{22}$$

with absorbing barrier at $x = a \geq 0$ and repulsive barrier at $x = b > a$ and initial value x is

$$T(x) = 2\int_a^x \frac{dy}{\psi(y)} \int_y^b \frac{\psi(z)dz}{B(z)} \tag{23}$$

where

$$\psi(x) = \exp\left(\int_a^x \frac{2A(x')dx'}{B(x')}\right). \tag{24}$$

In our case, both the drift and diffusion term are constant: $A(x) = A = -\alpha q$ and $B(x) = B = \frac{\sigma^2}{k}$. The computation of the integral proceeds as follows

$$\psi(x) = \exp\left(\frac{2A(x - a)}{B}\right)$$

$$\int_y^b \frac{\psi(z)dz}{B(z)} = \frac{1}{2A}[\exp(2A(b - a)/B) - \exp(2A(y - a)/B)]$$

$$2\int_a^x \frac{dy}{\psi(y)} \int_y^b \frac{\psi(z)dz}{B(z)} = 2\int_a^x \frac{1}{2A}\exp(2A(y - a)/B)^{-1}[\exp(2A(b - a)/B) - \exp(2A(y - a)/B)] =$$

$$= \frac{1}{A}\exp(2A(b - a)/B)\int_a^x \exp(-2A(y - a)/B) - \frac{1}{A}\int_a^x dy$$

We finally obtain

$$T(x) = \frac{B}{2A^2}\exp(2A(b - a)/B)[1 - \exp(-2A(x - a)/B)] - \frac{x - a}{A} \tag{25}$$

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In the context of our model it is $a = 0$, $b = 1$. Moreover, we assume $x = 1$, which implies that firms are created with the highest robustness. This is a conservative hypothesis with respect to the result which we will obtain. Substituting for the values of a, b, A, B, we obtain

$$T_f = \frac{1}{\alpha q} + \frac{\sigma^2}{2\alpha^2} \frac{(\exp(-2\alpha kq/\sigma^2) - 1)}{kq^2} \tag{26}$$

Bearing in mind that $q = q(k)$, the derivative of the mean first passage time with respect to the degree k can now be computed as follows. We denote the derivative of a function f equivalently as $D_k f$ or f',

$$T_f' = \frac{-q'}{\alpha q^2} + \frac{\sigma^2}{2\alpha^2 k^2 q^4}[exp(\frac{-2\alpha kq}{\sigma^2})(\frac{-2\alpha}{\sigma^2})D_k[kq]kq^2 - (exp(\frac{-2\alpha kq}{\sigma^2}) - 1)D_k[kq^2]] =$$

$$= \frac{-q'}{\alpha q^2} + \frac{\sigma^2}{2\alpha^2 k^2 q^4}[exp(\frac{-2\alpha kq}{\sigma^2})(\frac{-2\alpha}{\sigma^2})(q + kq')kq^2 - (exp(\frac{-2\alpha kq}{\sigma^2}) - 1)(q^2 + 2kqq') =$$

$$= \frac{-q'}{\alpha q^2} + \frac{\sigma^2}{2\alpha^2 k^2 q^4}\left(f_0(\frac{-2\alpha}{\sigma^2}(q + kq')kq^2 - (q^2 + 2kqq')) + (q^2 + 2kqq')\right) =$$

$$= \frac{-q'}{\alpha q^2} + \frac{\sigma^2}{2\alpha^2 k^2 q^4}\left(-f_0(f_1 + f2) + f2\right) = \tag{27}$$

where

$$\Phi_1 = \Phi(-\epsilon) \quad \Phi_2 = \Phi(\frac{\alpha\sqrt{k}}{\sigma} - \epsilon) \tag{28}$$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} exp(-t^2/2)dt \tag{29}$$

$$\Phi_2' = \frac{\alpha k^{-1/2}}{2\sigma\sqrt{2\pi}}exp\left(-\frac{(\alpha\sqrt{k}/\sigma - \epsilon)^2}{2}\right) \tag{30}$$

$$q = \frac{\Phi_1}{1 - \Phi_2 + \Phi_1} \tag{31}$$

$$q' = \frac{\Phi_1 \Phi_2'}{(1 - \Phi_2 + \Phi1)^2} \tag{32}$$

$$f_0 = exp\left(\frac{-2\alpha kq}{\sigma^2}\right) \tag{33}$$

$$f_1 = \frac{2\alpha}{\sigma^2}(q + kq')kq^2 \tag{34}$$

$$f_2 = q^2 + 2kqq' \tag{35}$$

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In order to determine in general the existence and the value of the maxima of $T(k)$, we solve the equation $T(k) = 0$, with $T(k)'' < 0$. Since the expression contains the error function, the solution of the equation is necessarily numerical.

We fix the parameter $\epsilon = 1$ and we vary (σ, α) in $[0.05, 1] \otimes [0.05, 1]]$, with step size of 0.005. The results were unaffected by using finer resolutions. For each pair (σ, α) we determine the values of k where $T'(k)$ changes from positive to negative sign. The idea is illustrated in Figure 4. The functional dependency of T and T' on k is shown for a particular choice of the parameters, displaying a non-monotonic behavior. In particular, in the inset, T' crosses the value of zero from above in the point of maximum. The general result is illustrated in Figure 5 (left). For each pair (σ, α) we find a unique maximum. The gray level indicates whether the maximum of T occurs at $k = 1$ or at $\infty > k > 1$. As it can be seen, $\alpha \leq \sigma$ is a sufficient condition to have existence and uniqueness of a maximum at $k > 1$. Figure 5 (right) shows the value of the optimal degree as a function of σ, α. This is typically smaller than thirty but it grows fast to larger values if the intensity α of the financial accelerator is very small.

The implication of the result are further illustrated in Figure 1. For a fixed σ, the dependence of T_f on k shows a point of maximum for each value of α. The existence of a maximum for T_f implies a minimum of the probability P_f of failure.
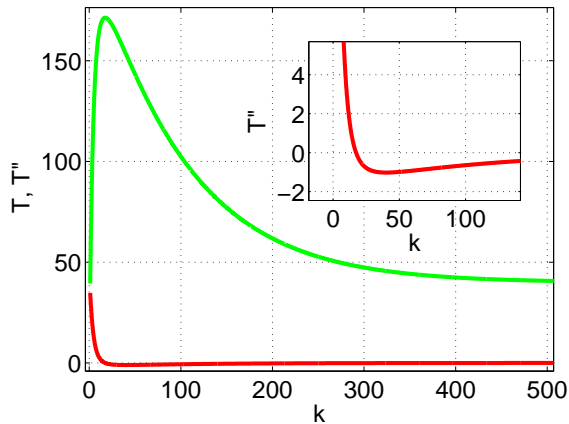

Figure 4: Dependency on on the degree k of the mean first passage time $T(k)$ and its derivative $T(k)'$. Parameter values: $\alpha = 0.025$, $\epsilon = 1$, $\sigma = 0.15$

□

Proof. Proof of Proposition 2 The specific cascading process in Equation (11) is modeled as follows:

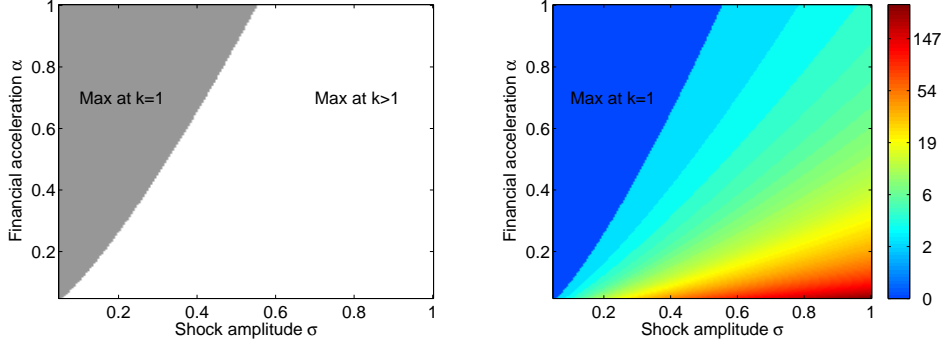$$\rho_i(\tau + 1) = \rho_i(\tau) - \frac{a}{k_i} \sum_{j=1}^{k} \delta_j(\tau) \tag{36}$$

Figure 5: Left: Number of maxima of $T(k)$. For each pair of values σ, α the color code illustrates the number of values of k where the derivative $T'(k)$ changes from positive to negative values. Right:Value of k in the point of maxima of $T(k)$ for each pair of values of the parameters σ, α.

where $\delta_j(\tau) = 1$ iff j fails at time τ. An equivalent but more convenient expression is

$$\rho_i(\tau + 1) = \rho_i(0) - \frac{a}{k_i} \sum_{j=1}^{k} \delta_j^c(\tau) \tag{37}$$

where $\delta_j^c(\tau) = 1$ iff j has failed at any time in $[0, \tau]$. We ask what is the *cumulative* fraction of nodes that have failed so far at time τ. This is

$$n(\tau + 1) = Pr\left\{\rho_i(\tau + 1) < \theta\right\} = Pr\left\{\sum_{j=1}^{k} \delta_j^c(\tau) > \rho_i(0)\right\} \tag{38}$$

For simplicity, we assume the network is a regular graph with degree k. Whether a given node i fails by the time step $\tau + 1$ depends on the number k_f of the neighbors that have already failed, out of the total number k of neighbors. At time step $\tau + 1$, the possible events are $k_f = \sum_{j=1}^{k} \delta_j^c(\tau) = 1, 2, ..., k$. In each of these events, the probability that node i fails, depends on the initial value of its robustness. Assume all failures that have occurred so far are uncorrelated across agents. Then, they follow a binomial distribution, $Pr\{\ k_f$ failures among k neighbors $\} = \binom{k}{k_f} p^{k_f} (1-p)^{k-k_f}$, where p is the probability that any given node has failed so far. In the limit of a large network it is $p = n(\tau)$. Finally, we need to take into account that firms do not recover during the cascade and thus the fraction of failure can only increase. Therefore,

$$n(\tau + 1) = \max\left\{s_0, \sum_{k_f=1}^{k} \binom{k}{k_f} n(\tau)^{k_f} (1 - n(\tau))^{k-k_f} Pr\left\{\rho_i(\tau) \leq \frac{ak_f}{k}\right\}\right\} \tag{39}$$

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where s_0 is the initial fraction of failures. This is a recursive equation of the type $n(\tau+1) = F(n(\tau))$. Once the probability distribution of ρ is specified, the fixed points are the solutions of $n = F(n)$. For example, in the case of regular graph, with ρ following a Gaussian distribution with mean μ and standard deviation σ we obtain:

$$n(\tau + 1) = \max\left\{ s_0, \sum_{k_f=1}^{k} \binom{k}{k_f} n(\tau)^j (1 - n(\tau))^{k-k_f} \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{ak_f/k} \exp(-\frac{(u-\mu)^2}{2\sigma^2}) \right\} = \tag{40}$$

$$= \max\left\{ s_0, \sum_{k_f=1}^{k} \binom{k}{k_f} n(\tau)^{k_f} (1 - n(\tau))^{k-k_f} \frac{1}{2}(1 + \mathrm{erf}(\frac{ak_f/k - \mu}{\sigma\sqrt{2}})) \right\} \tag{41}$$

$\square$

Notice that $F(n) \geq s_0$ with $s_0 > 0$ strictly for $\mu < \infty$ and $\sigma > 0$. Since in addition $F(n)$ is non decreasing, there exists at least one stable fixed point. There maybe more than one but what matters here is only the smallest stable fixed point s with $s \geq s_0$. The Equation above can be solved numerically with arbitrary precision for any choice of the parameters. The results are shown in Figure 2.

In general it would also be possible to account for heterogeneous degree distribution. This requires however a more extended analytical treatment that goes beyond the objective of this paper.

Proof. Proof of Proposition 3

Let us fix the amplitude σ of the shocks. The probability P^c that a fraction c of the agents fail in the same period, possibly due to a cascade, can be computed as follows. We use the same general result on mean first passage time described in proof of Proposition 2, Eq. (23). This time, we are interested in the first passage time at the value $\rho = m$ starting from the upper barrier $\rho = b = 1$.

$$T(k, m) = 2 \int_m^b \frac{dy}{\psi(y)} \int_y^b \frac{\psi(z)dz}{B(z)} \tag{42}$$

Substituting for $a = m$ and $b = 1$ in Equation (25) we obtain,

$$T_f = \frac{1-m}{\alpha q} + \frac{\sigma^2}{2\alpha^2} \frac{(\exp(-2\alpha kq(1-m)/\sigma^2) - 1)}{kq^2} \tag{43}$$

The probability that over time the robustness following Eq. (8) reaches the value m is given by the $\frac{1}{T(k,m)}$. In a mean field approach we take this as the probability that the average robustness $\bar{\rho}$ in the system reaches the value m.

On the other hand, the solution $S(k, m)$ of Eq. (12) gives the expected size of the cascade occurring when the average robustness in the system reaches the value m. We

are interested now in the mean first passage time $T^c(k)$ to reach the point where the occurring cascade is larger than c:

$$T^c(k) = \min_{m} \{T(k, m) | S(k, m) \geq c\} \tag{44}$$

Consequently, $P^c(k) = \frac{1}{T^c(k)}$ is the probability of occurrence of a cascade larger than c, which is a possible measure of systemic risk. The function of $T^c(k)$ in Equation (44) cannot be expressed in closed form. However, it can be computed numerically with arbitrary precision for any choice of the parameters α, σ, ϵ. Figure 2 (Left) shows the dependence of T^c on k for a particular choice of the parameters. The curve displays a maximum, which implies a minimum for the probability $P^c(k)$.

Differently from what done in the proof of Proposition 1, here there is no closed-form expression for the derivative of $T^c(k)$, so we compute numerically the point of maximum directly on the function $T^c(k)$. We do so for varying α in the interval $[0.001, 0.1]$ with steps of width 0.001, and for fixed values $\epsilon = 1$ and $\sigma = .4$ and $c = 0.25$. Notice that as discussed in Section 4.2 as a comment to the Proposition, the result is very robust with respect to c in virtue of the two regimes of cascades. As an illustration of the procedure, in Figure 3 (Right) the values of T^c are plotted as a function of k and α in the range of interest. It is apparent that for all values of α in the interval there is one global maximum of the expected time to large cascades T^c as a function of the degree k of risk diversification. In other words, there is a value of optimal connectivity degree as a function of α, which grows for decreasing values of α.

$\square$

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